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Some Properties of Higher Spin Rest-Mass Zero Fields in General Relativity

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Abstract

The propagation of a zero rest-mass test field of arbitrary spin $s > 1$ through curved space-time is found to be subject to strong constraints. A null test field is shown to be possible only in a restricted class of spaces previously introduced by Kundt and Thompson. This result is in fact a simultaneous generalization of the theorems of Robinson and of Goldberg and Sachs. For test fields of spin-2 in vacuum spaces, solutions of the propagation equation are restricted, save in a few exceptional cases, to constant multiples of the Weyl spinor. The exceptional cases are discussed, and appear to be physically uninteresting.

1. *Introduction*

The propagation equation for a zero rest-mass field of arbitrary spin $s > 0$ in a curved space time is (Penrose, 1965)

$$\nabla^{A\dot{X}} \Phi_{AB,\ldots,M} = 0 \tag{1.1}$$

where $\Phi_{AB,\ldots,M}$ is a totally symmetric spinor with $2s$ indices. The symbol $\nabla^{A\dot{X}}$ indicates covariant differentiation. Included in this formalism are the familiar Weyl neutrino field ($s = \frac{1}{2}$), electromagnetic field ($s = 1$), and gravitational field ($s = 2$). In the latter two cases equation (1.1) gives the spinor forms of Maxwell's equations and the vacuum Bianchi identities, respectively.

The concept of a *test* electromagnetic field is a familiar one in general relativity (Robinson, 1961). Such a field satisfies the source-free Maxwell's equations but does not contribute through its stress-energy tensor to the curvature of space. The above formalism clearly allows a generalization to the concept of a test spin-s field. Indeed, as no method exists for constructing an energy stress tensor for fields of spin $s > 1$, such fields must necessarily be regarded as test fields. The difficulties encountered with such fields in general relativity appears, moreover, in the existence of further constraint equations which any such field must satisfy (Buchdahl, 1958, 1962; Penrose, 1967), making impossible the proper specification of the null initial value problem. In fact, the very existence of any solution at all of equation (1.1) for $s > 1$ places a considerable restriction on the geometry of the underlying space.

In Section 2 we study the nature of this restriction in the case that equation (1.1) has a spin-s solution of null type. The existence of such solutions is shown to be equivalent both to the existence of a shear-free null geodesic congruence of rays and to the condition that the underlying space be algebraically special. In this way, we generalize simultaneously the theorems of Robinson (1961) and of Goldberg & Sachs (1962). Our theorem proves to be a new formulation of the generalization of the latter theorem due to Kundt & Thompson (1962).

The remainder of the paper concerns the existence of solutions (not necessarily null) of the spin-2 equation. In algebraically special vacuum spaces of type II and D the only solutions of the equation turn out to be constant multiples of the Weyl spinor itself and null-type solutions propagating along the repeated principal null directions. In algebraically general vacuum spaces (type I), we show that in 'almost every' case the only solutions are constant multiples of the Weyl spinor. The exceptional cases are also discussed, and it is shown that at most two independent solutions of the spin-2 equation can exist. This theorem is especially interesting, as it indicates that in a physically interesting situation the Bianchi identities uniquely determine the gravitational field (Weyl tensor) to within multiplication by a constant. This appears to be a kind of curved space generalization of a result in the linearized theory that, for bounded sources, if interest is restricted to retarded solutions alone then solving the Bianchi identities is equivalent to solving the linearized Einstein field equations (Szekeres, 1971). It would appear that the word 'retarded' can be omitted in the full theory, since 'advanced' solutions of the Bianchi identities cannot exist at all except in very restrictive circumstances. Yet another interpretation of this result is to say that the only zero rest-mass spin-2 field allowed in nature is gravity itself.

2. *The Generalized Robinson Theorem*

A spin-s field $\Phi_{AB,\ldots,M}$ is said to be of *null type* if it may be written in the form

$$\Phi_{AB,\ldots,M} = \Phi o_A o_B \ldots o_M \qquad (\Phi \neq 0)$$

for some 2-spinor field o_A. Equation (1.1) then takes the form

$$(\nabla^{A\dot{X}}\Phi) o_A o_B \ldots o_M + \Phi o_B o_C \ldots o_M(\nabla^{A\dot{X}} o_A) + \Phi o_A(\nabla^{A\dot{X}} o_B) o_C \ldots o_M + \cdots + \Phi o_A o_B \ldots (\nabla^{A\dot{X}} o_M) = 0 \qquad (2.1)$$

Introducing a second spinor field ι^A satisfying

$$o_A \iota^A = 1$$

so that o^A, ι^A together make up a basis for spin space, and contracting (2.1) first with $\iota^B \iota^C, \ldots, \iota^M$ and then with $o^B \iota^C, \ldots, \iota^M$ one has (for $s \geqslant 1$)

$$\kappa\Phi = 0 \tag{2.2}$$

$$\sigma\Phi = 0 \tag{2.3}$$

$$D\Phi = (\rho - 2s\epsilon)\Phi \tag{2.4}$$

$$\delta\Phi = (\tau - 2s\beta)\Phi \tag{2.5}$$

where $\kappa, \sigma, \rho, \tau, \epsilon$ and β are the standard spin coefficient symbols (Newman & Penrose, 1962) formed from the dyad o^A, ι^A (see Appendix). Thus, if a null-type solution of equation (1.1) exists, then

$$\kappa = \sigma = 0$$

i.e. o^A, or more particularly the null vector field l^μ which corresponds to $o_A \bar{o}_{\dot{B}}$, is geodesic and shear-free,

Conversely let us suppose there exists a geodesic shear-free null congruence, $\kappa = \sigma = 0$. We want to find under what conditions it is possible to integrate equations (2.4) and (2.5). Choosing a coordinate system such that

$$l^\mu = \delta_4{}^\mu, \qquad D \equiv \partial/\partial x^4,$$

there clearly exists a solution $\Phi = \Phi_0$ of equation (2.4). Any other solution of (2.4) is given by $\Phi = A\Phi_0$, where A is an arbitrary function of x^i $(i = 1, 2, 3)$.

Define

$$J = \delta\Phi - (\tau - 2s\beta)\Phi \tag{2.6}$$

Then applying the operator D to this equation, using (2.4) and the commutator and field equations given in the Appendix, the following equation emerges for J

$$DJ = (\rho + \bar{\rho} + (1 - 2s)\epsilon - \bar{\epsilon})J + (2s - 2)\Psi_1 \Phi \tag{2.7}$$

It is, however, always possible to pick Φ such that $J = 0$ on the initial hypersurface $x^4 = 0$ for this amounts simply to solving a single complex partial differential equation within this hypersurface

$$\Phi_0 \delta \log A + J_0 = 0$$

where J_0 is the value of J obtained from (2.6) on putting $\Phi = \Phi_0$.

If $s = 1$, equation (2.7) is homogeneous in J and if $J = 0$ initially, it must remain zero. Hence, Φ satisfies both (2.4) and (2.5). This proves Robinson's original theorem.

When $s > 1$, equation (2.7) is homogeneous if and only if $\Psi_1 = 0$, i.e. if and only if the space is algebraically special, since $\Psi_0 = 0$ follows automatically from $\kappa = \sigma = 0$ (see Appendix). We have then the following:

Generalized Robinson Theorem: Any two of the following implies the third

(α) *There exists a geodesic shear-free congruence,*
(β) *The Weyl tensor is algebraically special,*
(γ) *There exists a null-type solution of the spin-s rest-mass zero equation* (1.1) *for every* $s > 1$.

In fact the theorem we have proved is rather stronger than this. If (γ') is the condition

(γ') *There exists a null-type solution of the spin-s rest-mass zero equation* (1) *for some* $s > 1$,

we have shown that

$$(\gamma') \Rightarrow (\alpha) \Rightarrow (\beta) \Rightarrow (\gamma)$$

Thus the existence of a null-type solution of equation (1.1) for some $s > 1$ is enough to ensure the validity of the Goldberg-Sachs theorem (Goldberg & Sachs, 1962; Newman & Penrose, 1962). The class of spaces in which the Goldberg-Sachs theorem holds has been determined by Kundt & Thompson (1962). Our theorem gives a new characterization of these spaces as the class of spaces in which there exists a null-type solution of equation (1.1) for some $s > 1$.

3. *The Constraint Equation*

Given a spin-s ($s \geqslant 3/2$) zero rest-mass field $\Phi_{AB,\ldots,M}$ satisfying equation (1.1), the Ricci identity may be used (Penrose, 1967) to deduce an *algebraic* identity

$$\Psi^{ABC}{}_{(D}\Phi_{E,\ldots,M)ABC} = 0 \tag{3.1}$$

This equation places strong constraints on the possible values of $\Phi_{AB,\ldots,M}$. In the case of a spin-2 field Φ_{ABCD} equation (3.1) takes the following form when written out in terms of a spin basis o_A, ι_A (see Appendix):

$$\Psi_0\Phi_3 - 3\Psi_1\Phi_2 + 3\Psi_2\Phi_1 - \Psi_3\Phi_0 = 0 \tag{3.1a}$$

$$\Psi_0\Phi_4 - 2\Psi_1\Phi_3 + 2\Psi_3\Phi_1 - \Psi_4\Phi_0 = 0 \tag{3.1b}$$

$$\Psi_1\Phi_4 - 3\Psi_2\Phi_3 + 3\Psi_3\Phi_2 - \Psi_4\Phi_1 = 0 \tag{3.1c}$$

We consider these relations firstly in the case that the metric is algebraically special. This means that there exists a repeated principal spinor which may be taken to be o_A, so that $\Psi_0 = \Psi_1 = 0$. Separate cases arise corresponding to the different possible Petrov types.

Petrov type II. The spinor ι_A may be chosen so that $\Psi_2 \neq 0$, $\Psi_3 = 0$, $\Psi_4 \neq 0$. Equations (3.1a)–(3.1c) give

$$\Phi_1 = \Phi_0 = \Phi_3 = 0$$

i.e.

$$\Phi_{ABCD} = \alpha \Psi_{ABCD} + N^{(1)}_{ABCD} \tag{3.2}$$

where

$$N^{(1)}_{ABCD} = \beta o_A o_B o_C o_D$$

and α and β are scalar functions.

Petrov type D. ι_A may be chosen so that $\Psi_2 \neq 0$, $\Psi_3 = \Psi_4 = 0$. Equations (3.1a)–(3.1c) give

$$\Phi_1 = \Phi_3 = 0$$

i.e.

$$\Phi_{ABCD} = \alpha \Psi_{ABCD} + N^{(1)}_{ABCD} + N^{(2)}_{ABCD} \tag{3.3}$$

where

$$N^{(2)}_{ABCD} = \gamma o_A o_B o_C o_D$$

Petrov type III. $\Psi_2 = 0$, $\Psi_3 \neq 0$. Equations (3.1a)–(3.1c) give

$$\Phi_0 = \Phi_1 = \Phi_2 = 0$$

i.e. Φ_{ABCD} is of type III or N.

Petrov type N. $\Psi_2 = \Psi_3 = 0$. Equations (3.1a)–(3.1c) give

$$\Phi_0 = \Phi_1 = 0$$

i.e. Φ_{ABCD} is algebraically special.

In types II and D, it may be shown that if Ψ_{ABCD} satisfies the vacuum Bianchi identities,

$$\nabla^{A\dot{Y}} \Psi_{ABCD} = 0 \tag{3.4}$$

then α must be a constant [a space satisfying the condition (3.4) has previously been termed a *C*-space by one of us (Szekeres, 1963)]. For, substituting (3.2) and (3.3) into equation (1.1) give, respectively,

$$0 = \Psi_{ABCD} \nabla^{D\dot{Y}} \alpha + \nabla^{D\dot{Y}} N^{(1)}_{ABCD}$$

and

$$0 = \Psi_{ABCD} \nabla^{D\dot{Y}} \alpha + \nabla^{D\dot{Y}} N^{(1)}_{ABCD} + \nabla^{D\dot{Y}} N^{(2)}_{ABCD}$$

Contracting these equations first with $o^A o^B \iota^C$ and then with $o^A \iota^B \iota^C$ and using the shear-free geodesic condition

$$o^A o_D \nabla^{D\dot{Y}} o_A = 0 \Leftrightarrow \kappa = \sigma = 0$$

which follows immediately from (3.4), yields

$$o_D \nabla^{D\dot{Y}} \alpha = \iota_D \nabla^{D\dot{Y}} \alpha = 0 \Leftrightarrow \nabla^{D\dot{Y}} \alpha = 0$$

and

$$\nabla^{D\dot{Y}} N^{(1)}_{ABCD} = \nabla^{D\dot{Y}} N^{(2)}_{ABCD} = 0$$

Hence α is a constant, and the null fields $N^{(1)}$ and $N^{(2)}$ satisfy the rest-mass zero spin-2 equation. (To show that $N^{(2)}$ satisfies the equation in type D it is necessary to also use the fact that ι^A is shear-free and geodesic.)

Thus in a C-space Φ_{ABCD} is uniquely determined by Ψ_{ABCD} to within a constant factor and null-type fields propagating along the repeated null direction(s). The latter always exist by our generalized Robinson theorem.

When Ψ_{ABCD} is of type III, Φ_{ABCD} must also be of type III, and if Ψ_{ABCD} is of type N then Φ_{ABCD} must be algebraically special. It is not possible to obtain any further uniqueness conditions on Φ_{ABCD} in these cases.

4. *The Algebraically General Case*

If the metric is algebraically general, o_A and ι_A could be chosen to be principal spinors so that $\Psi_0 = \Psi_4 = 0$, $\Psi_1 \neq 0$, $\Psi_3 \neq 0$ (called a *principal dyad*). To investigate equations (3.1a)–(3.1c) it will, however, turn out more useful to adopt another dyad having the property that

$$\Psi_1 = \Psi_3 = 0, \qquad \Psi_0 \neq 0, \qquad \Psi_4 \neq 0 \tag{4.1}$$

The existence of such a dyad is easily shown by applying the following unimodular transformation to any principal dyad o_A, ι_A:

$$\begin{aligned} o_A' &= (\Psi_3)^{1/2} o_A + (\Psi_1)^{1/2} \iota_A \\ \iota_A' &= -\tfrac{1}{2}(\Psi_1)^{-1/2} o_A + \tfrac{1}{2}(\Psi_3)^{-1/2} \iota_A \end{aligned} \tag{4.2}$$

It is further possible by a scaling transformation

$$o_A'' = a o_A', \qquad \iota_A'' = a^{-1} \iota_A'$$

to achieve that

$$\Psi_0 = \Psi_4 \neq 0 \tag{4.3}$$

(if either Ψ_0 or Ψ_4 vanish the metric is clearly algebraically special). A dyad satisfying (4.1) and (4.3) will be called an *interaction dyad* [such dyads appear naturally in the collisional interaction of plane waves (Szekeres, 1972)]. There will in general be six different interaction dyads, corresponding to six different choices of principal dyad on which to perform the unimodular transformation (4.2). An exceptional case occurs when

$$\Psi_0 = \pm 3\Psi_2$$

in which case the metric is algebraically special (type D) having repeated principal spinors $o_A \pm \iota_A$ (for $\Psi_0 = +3\Psi_2$), $o_A \pm i\iota_A$ (for $\Psi_0 = -3\Psi_2$). There is then a continuous one-parameter family of interaction dyads.

In an interaction dyad equations (3.1a)–(3.1c) read

$$\begin{aligned} \Psi_0 \Phi_3 + 3\Psi_2 \Phi_1 &= 0 \\ -3\Psi_2 \Phi_3 - \Psi_0 \Phi_1 &= 0 \\ \Psi_0(\Phi_4 - \Phi_0) &= 0 \end{aligned}$$

Hence in the algebraically general case ($\Psi_0 \neq \pm 3\Psi_2$), it follows that

$$\Phi_1 = \Phi_3 = 0, \qquad \Phi_0 = \Phi_4$$

and an interaction dyad for Ψ_{ABCD} is also an interaction dyad for Φ_{ABCD}.

The spin-2 equations (or vacuum Bianchi identities) read, in an interaction dyad,

$$
\begin{aligned}
D\Phi_0 &= -(4\epsilon - \rho)\,\Phi_0 - 3\lambda\Phi_2 \\
\Delta\Phi_0 &= (4\gamma - \mu)\,\Phi_0 + 3\sigma\Phi_2 \\
\delta\Phi_0 &= -(4\beta - \tau)\,\Phi_0 - 3\nu\Phi_2 \\
\bar{\delta}\Phi_0 &= (4\alpha - \beta)\,\Phi_0 + 3\kappa\Phi_2 \\
D\Phi_2 &= -\lambda\Phi_0 + 3\rho\Phi_2 \\
\Delta\Phi_2 &= \sigma\Phi_0 - 3\mu\Phi_2 \\
\delta\Phi_2 &= -\nu\Phi_0 + 3\tau\Phi_2 \\
\bar{\delta}\Phi_2 &= \kappa\Phi_0 - 3\pi\Phi_2
\end{aligned}
$$

Using

$$\phi_{,\mu} = n_\mu\, D\phi + l_\mu\, \Delta\phi - \bar{m}_\mu\, \delta\phi - m_\mu\, \bar{\delta}\phi$$

these equations may be written

$$\Phi_{2,\mu} = A_\mu\,\Phi_0 + 3B_\mu\,\Phi_2 \qquad (4.4a)$$

$$\Phi_{0,\mu} = (B_\mu + C_\mu)\,\Phi_0 + 3A_\mu\,\Phi_2 \qquad (4.4b)$$

where

$$
\begin{aligned}
A_\mu &= -\lambda n_\mu + \sigma l_\mu + \nu\bar{m}_\mu - \kappa m_\mu \\
B_\mu &= \rho n_\mu - \mu l_\mu - \tau\bar{m}_\mu + \pi m_\mu \\
C_\mu &= -4\epsilon n_\mu + 4\gamma l_\mu + 4\beta\bar{m}_\mu - 4\alpha m_\mu
\end{aligned}
$$

The integrability conditions for (4.4a) and (4.4b) are

$$\Phi_{0,[\mu\nu]} = \Phi_{2,[\mu\nu]} = 0$$

i.e.

$$0 = 3\Phi_2\, B_{[\mu,\nu]} + \Phi_0(A_{[\mu,\nu]} + A_{[\mu}(C_{\nu]} - 2B_{\nu]})) \qquad (4.5a)$$

$$0 = 3\Phi_2(A_{[\mu,\nu]} + A_{[\mu}(2B_{\nu]} - C_{\nu]})) + \Phi_0(B_{[\mu,\nu]} + C_{[\mu,\nu]}). \qquad (4.5b)$$

In vacuo, or in a C-space, this pair of linear equations for Φ_0, Φ_2 must admit a solution since the spin-2 equation has a non-trivial solution in the Weyl spinor. On the other hand

$$\Phi_0 = A\Psi_0, \qquad \Phi_2 = A\Psi_2 \qquad (A \text{ arbitrary}) \qquad (4.6)$$

will be the only possible solutions of (4.5a) and (4.5b) unless the matrix of coefficients vanishes identically, i.e. unless

$$B_{[\mu,\nu]} = C_{[\mu,\nu]} = A_{[\mu,\nu]} = 0 \qquad (4.7a)$$

and

$$A_{[\mu}C_{\nu]} = 2A_{[\mu}B_{\nu]} \qquad (4.7b)$$

Substituting (4.6) into (4.4a) and (4.4b), and using the fact that $\Phi_0 = \Psi_0$, $\Phi_2 = \Psi_2$ satisfy these equations, implies at once that

$$A_{,\mu} = 0$$

i.e.

$$A = \text{const.}$$

Hence, if (4.7a) and (4.7b) are not identically satisfied, the only solution of the rest-mass zero spin-2 equation is the Weyl spinor itself or constant multiples of it.

In the exceptional case that (4.7a) and (4.7b) are satisfied, there exist complex scalar functions a, b, c such that

$$A_\mu = a_{,\mu}, \qquad B_\mu = b_{,\mu}, \qquad C_\mu = c_{,\mu}$$

with

$$c - 2b = f(a)$$

Two cases may be distinguished:

(i) $a \neq 0$, $b \neq b(a)$.

Equations (4.4a) and (4.4b) imply that

$$\Phi_0 = \Phi_0(a, b), \qquad \Phi_2 = \Phi_2(a, b)$$

and

$$\partial\Phi_2/\partial a = \Phi_0, \qquad \partial\Phi_2/\partial b = 3\Phi_2$$
$$\partial\Phi_0/\partial a = 3\Phi_2 + f'(a)\,\Phi_0, \qquad \partial\Phi_0/\partial b = 3\Phi_0$$

Hence

$$\Phi_2 = F(a)\,e^{3b}, \qquad \Phi_0 = F'(a)\,e^{3b} \quad (' = d/da)$$

where $F(a)$ satisfies the linear second-order ordinary differential equation

$$F'' - f'\,F' - 3F = 0$$

This equation has two independent solutions $F_1(a)$, $F_2(a)$, and

$$\Phi_2 = e^{3b}(c_1 F_1(a) + c_2 F_2(a))$$
$$\Phi_0 = e^{3b}(c_1 F_1'(a) + c_2 F_2'(a))$$

where c_1 and c_2 are arbitrary constants.

(ii) $b = b(a)$, $c = c(a)$.

A similar argument shows that

$$\Phi_2 = \Phi_2(a), \qquad \Phi_0 = \Phi_0(a)$$

and

$$\Phi_2' = \Phi_0 + 3b'\,\Phi_2$$
$$\Phi_0' = \Phi_0(b' + c') + 3\Phi_2$$

a pair of ordinary differential equations having a general solution of the form

$$(\Phi_0, \Phi_2) = c_1(\Phi_0^{(1)}, \Phi_2^{(1)}) + c_2(\Phi_0^{(2)}, \Phi_0^{(2)}).$$

Thus in each case, when equations (4.7a) and (4.7b) hold there are at most two independent solutions of the spin-2 rest-mass zero equation.

$$\Phi_{ABCD} = c_1\,\Phi^{(1)}_{ABCD} + c_2\,\Phi^{(2)}_{ABCD}$$

One of these, say $\Phi^{(1)}_{ABCD}$, may be taken equal to the Weyl spinor Ψ_{ABCD} itself.

> *Theorem: In an algebraically general C-space, the Bianchi identities either have a unique solution to within constant multiples, or its solutions are linear combinations of at most two independent solutions.*

The strongest possible statement of this theorem in algebraically special cases has been made in the previous section. In that case it was always possible to add null type solutions in accordance with the generalized Robinson theorem. The theorem, as it stands may be extended to all non-vacuum cases but then the Bianchi identities and the spin-2 equation are not identical in general. The proof is similar, inhomogeneous terms constructed from Ricci tensor components appearing on the right-hand sides of equations (4.4a) and (4.4b).

In conclusion, it would be of interest to know whether spaces satisfying the exceptional conditions (4.7a) and (4.7b) exist at all.

Consider the metric

$$ds^2 = 2\exp[\tfrac{1}{2}(a^2-1)\,U]\,du\,dv - 2\{\exp[(1-a)\,U]\,dx^2 + \exp[(1+a)\,U]\,dy^2\}$$

where

$$U = \log(u+v)$$

This is a vacuum metric, one of a class of metrics derived by Landau & Lifschitz (1962), displayed here in null coordinates. An interaction tetrad is

$$l_\mu = \{\exp[\tfrac{1}{4}(1-a^2)\,U], 0, 0, 0\}$$
$$n_\mu = \{0, \exp[\tfrac{1}{4}(1-a^2)\,U], 0, 0\}$$
$$m^\mu = \tfrac{1}{2}\{0, 0, \exp[\tfrac{1}{2}(a-1)\,U],\, i\exp[-\tfrac{1}{2}(a+1)\,U]\}$$

in which the Weyl tensor has components

$$\Psi_0 = \Psi_4 = \frac{a(1-a^2)}{4}\exp[-\tfrac{1}{4}(3+a^2)\,U]$$
$$= -a\Psi_2, \qquad \Psi_1 = \Psi_3 = 0$$

Thus the metric is algebraically general except when $a = 0$ or ± 3 (type D), or $a = \pm 1$ (flat space). The vectors A_μ, B_μ, C_μ may be calculated to be

$$A_\mu = \tfrac{1}{2}aU_{,\mu}, \qquad B_\mu = -\tfrac{1}{2}U_{,\mu}, \qquad C_\mu = \tfrac{1}{2}(1-a^2)\,U_{,\mu}$$

Hence, conditions (4.7a) and (4.7b) are satisfied and the metric is of the exceptional type. The spin-2 rest-mass zero equation may be explicitly integrated, giving two independent solutions, one the Weyl tensor having components given above, the other having components

$$\Phi_0 = \Phi_4 = b = \text{const.}, \qquad \Phi_2 = \frac{ab}{3}$$

Most physically interesting metrics cannot, however, be expected to satisfy the very restrictive conditions (4.7a) and (4.7b), and a unique solution of the Bianchi identities is generally to be expected.

Appendix

A pair of spinors o^A, ι^A satisfying

$$o_A \iota^A = 1 \qquad \epsilon_{AB} = o_A \iota_B - \iota_A o_B$$

constitutes a basis (dyad) for spinors. Newman & Penrose (1962) introduced the generic symbol $\zeta_a{}^A$ $(a = 0, 1)$

$$\zeta_0{}^A = o^A, \qquad \zeta_1{}^A = \iota^A$$

and defined

$$\Gamma_{abcd} = \zeta_{aA;\mu} \zeta_b{}^A \sigma^\mu_{cd} = \Gamma_{bacd}$$

where

$$\sigma^\mu_{cd} = \sigma^\mu_{C\dot{D}} \zeta_c{}^C \bar{\zeta}_d{}^{\dot{D}}$$

The components of Γ_{abcd} are the *spin coefficients*,

$$\begin{array}{lll} \Gamma_{000\dot{0}} = \kappa & \Gamma_{010\dot{0}} = \epsilon & \Gamma_{110\dot{0}} = \pi \\ \Gamma_{001\dot{0}} = \rho & \Gamma_{011\dot{0}} = \alpha & \Gamma_{111\dot{0}} = \lambda \\ \Gamma_{000\dot{1}} = \sigma & \Gamma_{010\dot{1}} = \beta & \Gamma_{110\dot{1}} = \mu \\ \Gamma_{001\dot{1}} = \tau & \Gamma_{011\dot{1}} = \gamma & \Gamma_{111\dot{1}} = \nu \end{array}$$

In addition, intrinsic derivatives D, Δ, δ and $\bar{\delta}$ are defined by

$$D\phi = o^A \bar{o}^{\dot{D}} \nabla_{A\dot{D}} = l^\mu \nabla_\mu \phi$$
$$\Delta\phi = \iota^A \bar{\iota}^{\dot{D}} \nabla_{A\dot{D}} = n^\mu \nabla_\mu \phi$$
$$\delta\phi = o^A \bar{\iota}^{\dot{D}} \nabla_{A\dot{D}} = m^\mu \nabla_\mu \phi$$
$$\bar{\delta}\phi = \iota^A \bar{o}^{\dot{D}} \nabla_{A\dot{D}} = \bar{m}^\mu \nabla_\mu \phi$$

l^μ, n^μ, m^μ, $\bar{m}^\mu$ being the null tetrad associated with the dyad o^A, ι^A.

The spinor equivalent of the Riemann tensor $R_{\mu\nu\rho\sigma}$ decomposes as follows:

$$-R_{A\dot{E}B\dot{F}C\dot{G}D\dot{H}} = \Psi_{ABCD} \epsilon_{\dot{E}\dot{F}} \epsilon_{\dot{G}\dot{H}} + \epsilon_{AB} \epsilon_{CD} \bar{\Psi}_{\dot{E}\dot{F}\dot{G}\dot{H}}$$
$$+ 2\Lambda(\epsilon_{AC} \epsilon_{BD} \epsilon_{\dot{E}\dot{F}} \epsilon_{\dot{G}\dot{H}} + \epsilon_{AB} \epsilon_{CD} \epsilon_{\dot{E}\dot{H}} \epsilon_{\dot{F}\dot{G}})$$
$$\epsilon_{AB} \Phi_{CD\dot{E}\dot{F}} \epsilon_{\dot{G}\dot{H}} + \epsilon_{CD} \Phi_{AB\dot{G}\dot{H}} \epsilon_{\dot{E}\dot{F}}$$

where $\Psi_{ABCD} = \Psi_{(ABCD)}$ is determined uniquely from the Weyl tensor and $\Phi_{AB\dot{E}\dot{F}}$ and Λ from the Ricci tensor. The dyad components of the Weyl spinor Ψ_{ABCD} are denoted

$$\Psi_0 = \Psi_{0000}, \qquad \Psi_1 = \Psi_{0001}, \qquad \Psi_2 = \Psi_{0011},$$
$$\Psi_3 = {}_{0111}, \qquad \Psi_4 = \Psi_{1111}$$

where

$$\Psi_{abcd} = \Psi_{ABCD} \zeta_a{}^A \zeta_b{}^B \zeta_c{}^C \zeta_d{}^D$$

Among the field equations (Ricci identities) only the following are of relevance to this paper. They are given for the case $\kappa = \sigma = 0$ which is appropriate to Section 2.

$$\Psi_0 = 0$$
$$D\tau = (\tau + \bar{\pi})\rho + (\epsilon - \bar{\epsilon})\tau + \Psi_1 + \Phi_{01}$$
$$D\rho - \bar{\delta}\epsilon = (\bar{\rho} - \bar{\epsilon})\beta - (\bar{\alpha} - \bar{\pi})\epsilon + \Psi_1$$
$$\delta\rho = \rho(\bar{\alpha} + \beta) + (\rho - \bar{\rho})\tau - \Psi_1 + \Phi_{01}$$

where

$$\Phi_{01} = \Phi_{AB\dot{C}\dot{D}}\, o^A o^B \bar{o}^{\dot{C}} \bar{\iota}^{\dot{D}}$$

The commutator of the derivatives δ and D, also used in Section 2, is (again with $\kappa = \sigma = 0$)

$$(\delta D - D\delta)\phi = (\bar{\alpha} + \beta - \bar{\pi})D\phi - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta\phi.$$

The vacuum Bianchi identities

$$\nabla^{A\dot{D}}\Psi_{ABCD} = 0$$

are

$$D\Psi_1 - \bar{\delta}\Psi_0 = -3\kappa\Psi_2 + (2\epsilon + 4\rho)\Psi_1 - (4\alpha - \pi)\Psi_0$$
$$D\Psi_2 - \bar{\delta}\Psi_1 = -2\kappa\Psi_3 + 3\rho\Psi_2 - (2\alpha - 2\pi)\Psi_1 - \lambda\Psi_0$$
$$D\Psi_3 - \bar{\delta}\Psi_2 = -\kappa\Psi_4 - (2\epsilon - 2\rho)\Psi_3 + 3\pi\Psi_2 - 2\lambda\Psi_1$$
$$D\Psi_4 - \bar{\delta}\Psi_3 = -(4\epsilon - \rho)\Psi_4 + (4\pi + 2\alpha)\Psi_3 - 3\lambda\Psi_2$$
$$\Delta\Psi_0 - \delta\Psi_1 = +(4\gamma - \mu)\Psi_0 - (4\tau + 2\beta)\Psi_1 + 3\sigma\Psi_2$$
$$\Delta\Psi_1 - \delta\Psi_2 = \nu\Psi_0 + (2\gamma - 2\mu)\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3$$
$$\Delta\Psi_2 - \delta\Psi_3 = 2\nu\Psi_1 - 3\mu\Psi_2 + (2\beta - 2\tau)\Psi_3 + \sigma\Psi_4$$
$$\Delta\Psi_3 - \delta\Psi_4 = 3\nu\Psi_2 - (2\gamma + 4\mu)\Psi_3 + (4\beta - \tau)\Psi_4$$

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